Lecture 17 11/26

Recurrence Relations

Probability

Graph Theory

Recurrence Relations

Def A recurrence relation is an eqn that recursively defines a seq. or array

Each term is defined as a func of the preceding terms

Ex $n! = n \times (n-1)!$ (Factorial)

$F_n = F_{n-1} + F_{n-2}$ (Fibonacci numbers)

$a_{2n+1} = a_n + a_{n+1}$ with $a_{2n} = a_n$, (Stern's diatomic series)

$a_0 = 0,\ a_1 = 1$

$a_2 = a_1 = 1$

$a_3 = a_1 + a_2 = 1+1 = 2$

$a_4 = a_2 = 1$

$a_5 = a_2 + a_3 = 1+2 = 3$

Some Counting probs cannot be solved easily via simple combinatorics

Ex How many bit strings of length n can be formed that do not contain two consecutive zeros?

$n=1 \Rightarrow \left.\begin{matrix}0\\1\end{matrix}\right\} \Rightarrow$ 2 strings $\Rightarrow a_1 = 2$

$n=2 \Rightarrow \left.\begin{matrix}01\\10\\11\\\cancel{00}\end{matrix}\right\} \Rightarrow$ 3 strings $\Rightarrow a_2 = 3$

To generalize, note

$S_1 \subset S$ set of strings ending in 1

$S_0 \subset S$ " " " " 0

$S = S_1 \cup S_0$ is disjoint

$|S_1| + |S_2| = S$

$\Rightarrow a_{n+1} = a_n + a_{n-1}$

$a_3 = 2+3 = 5$

$a_4 = 3+5 = 8$

$a_5 = 5+8 = 13$

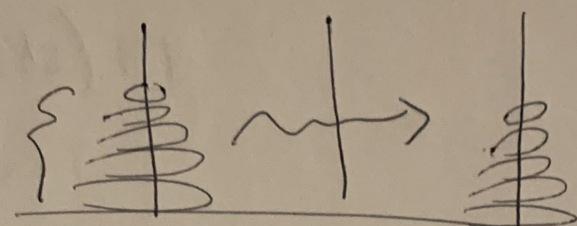
Another Fib Seq.

Ex Tower of Hanoi

- 3 poles
- n plates

start goal

n



1 2 3

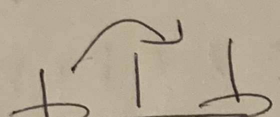
Rules

1) 1 plate per move
2) a bigger plate never above smaller plate

Can we find a recursive formula & turn into a closed formula?

⇒ Can prove by induction

when $n=1$

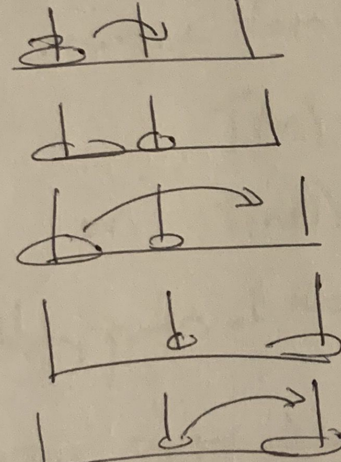


1 move

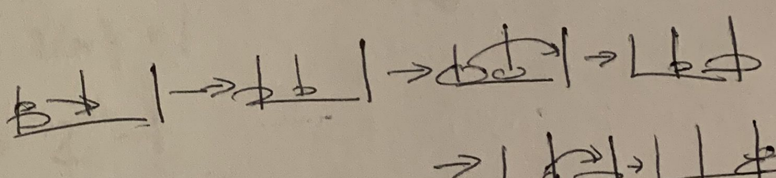
$T(1) = 1$

$T(n) = \#$ steps for n plates

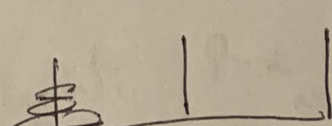
$n=2$



$T(2) = 3$ steps



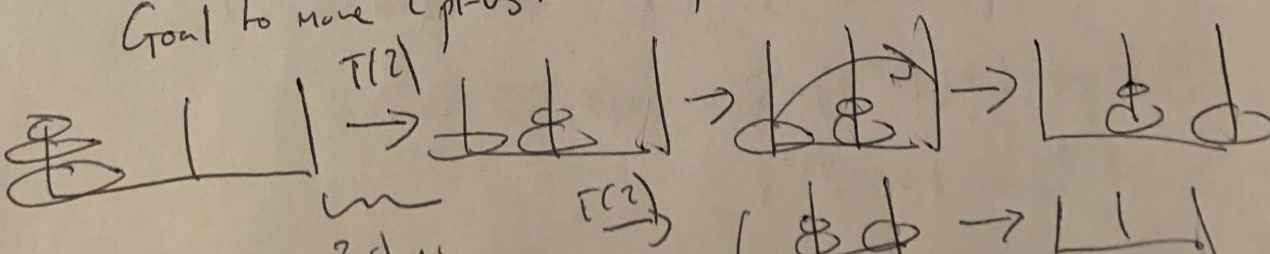
$n=3$



Lets exploit recursive structure of prob (pretend only have 2 plates)

Use strategy of $T(2)$ 2 times.

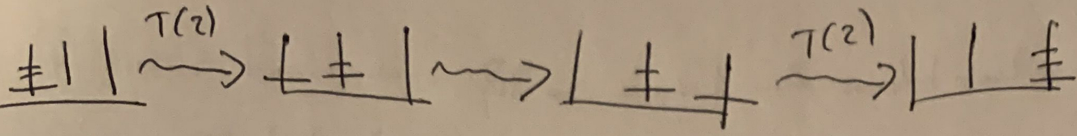
Goal to move 2 plates to second poll



$T(2)$

3 steps

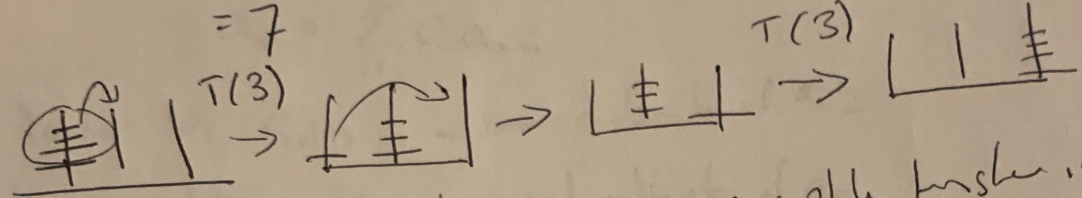
$T(2)$

n=3 

$T(3) = 2 \times T(2) + 1$

$= 2 \times 3 + 1$

$= 7$

n=4 

Imagine bottom plate isn't there, pick up upper plate transfer it

$T(4) = 2 \times T(3) + 1$

$= 2 \times 7 + 1 = 15$

What is recursive formula?

n	1	2	3	4	5
$T(n)$	1	3	7	15	31

$T(n) = 2 \times T(n-1) + 1$

or $T(n+1) = 2 \times T(n) + 1$

Always adding powers of two $\Rightarrow 2^n - 1$

Hyp: $T(n) = \cancel{2 \times T(n-1)} \; 2^n - 1$

Proof by Ind: Base $n=1 \Rightarrow 2^n - 1 = 2^1 - 1 = 1$ ✓

IH assume $n = k$ true $\Rightarrow 2^k - 1 = T(k)$

IS $n = k+1$ true: $2^{k+1} - 1 = T(k+1)$

$T(k+1) = 2 \times T(k) + 1 = 2 \times (2^k - 1) + 1$ by IH

$= 2^{k+1} - 2 + 1 = 2^{k+1} - 1$ ✓

Def A linear homogeneous RR of degree k with constant coefficients is an RR of the form

$$a_n = C_1 a_{n-1} + C_2 a_{n-2} + \cdots + C_k a_{n-k}$$

$$a_n = \sum_{i=1}^{k} C_i a_{n-i}$$

linear b/c RHS is a linear combination of a_i's

homogeneous b/c no terms are not multiples of a_k

Coeffs are constant, degree is k

By PMI

A seq $\{a_i\}_{i=1}^{n}$ satisfying above uniquely defined by the RR and the k initial conditions

Solving LHRR

Ex: $a_n = 5a_{n-1} - 6a_{n-2}$ $n \geq 2$ $a_0 = 1$, $a_1 = 0$

Idea: Turn RR into algebraic eqn (characteristic eqn)

Note that the LHRR has deg 2 $\Rightarrow$ 2^{nd} degree polynomial

$x^2 = 5x - 6 \Rightarrow x^2 - 5x + 6 = 0$

$(x-2)(x-3) = 0 \Rightarrow x = 2, 3$

if distinct real roots, $\Rightarrow a_n = C_1 \times 2^n + C_2 \times 3^n$

if $(x-2)(x-2) \Rightarrow$ repeated root of multiplicity 2

$a_n = C_1 \cdot 2^n + n \cdot C_2 \cdot 2^n$

$1 = C_1 \cdot 2^0 + C_2 3^0$

$1 = C_1 + C_2$

$0 = C_1 \cdot 2 + C_2 \cdot 3$

System of eqns

$-2 = -2C_1 + -2C_2$

$0 = 2C_1 + 3C_2$

$\Rightarrow -2 = C_2$

$C_1 = 3$

$= a_n = 3 \times 2^n + 2 \cdot 3^n$

Ex $a_n = 3a_{n-1}$

$a_0 = 7$ $\Rightarrow a_n = 7 \times 3^n$

$a_n = d \cdot a_{n-1}$, $a_0 = k$

$a_n = k \cdot (d)^n$

or $a_n - a_{n-1} = K$ $a_0 = C$

$a_n = C + \sum_{i=1}^{n} K$

1. Char poly
2. Factor
3. Determine form of a_n
4. Solve for coefficients
   - systems of eqns
   - matrices

Ex. $a_n - a_{n-1} - 6a_{n-2} = 0$ $a_0 = 1$

$a_1 = 8$

$x^2 - x - 6 = 0$

$(x-3)(x+2) = 0$

$x = -2, 3 \Rightarrow$ $a_n = \alpha(-2)^n + \beta(3)^n$

when $n = 0 \Rightarrow a_0 = 1 = \alpha(-2)^0 + \beta(3)^0$

$n = 1 \Rightarrow a_1 = 8 = \alpha(-2)^1 + \beta(3)^1$

$1 = \alpha + \beta$ $\Rightarrow$ $2 = 2\alpha + 2\beta$

$8 = -2\alpha + 3\beta$ $\quad$ $8 = -2\alpha + 3\beta$

$10 = 5\beta$

$\beta = 2$ $\Rightarrow \alpha = -1$

$$\boxed{a_n = -1(-2)^n + 2(3)^n}$$

$\beta = 1/2$ $\alpha = 1$

Ex $a_n - 4a_{n-1} + 4a_{n-2} = 0$ $a_0 = 1, a_1 = 3$

$2^n + \frac{1}{2} n 2^n$

$\Rightarrow a_n = \alpha(2^n) + \beta \cdot n(2^n) \Rightarrow$ $= a_n = 2^n(1 + \frac{1}{2}n)$

When roots same

$r = \underbrace{3,3,3\ldots,3}_{K}, \underbrace{2,2,2\ldots 2}_{M}$

$$a_n = \alpha_1(3^n) + \alpha_2 n(3^n) + \cdots + a_K n^K(3^n)$$
$$+ \beta_1(2^n) + \beta_2 n(2^n) + \cdots \beta_M n^M(2^n)$$

Thm Let $c_1, c_2 \in \mathbb{R}$. Suppose $r^2 - c_1 r - c_2 = 0$ has two distinct roots r_1, r_2. The seq $\{a_n\}$ is a sol of RR $a_n = c_1 a_{n-1} + c_2 a_{n-2}$ iff $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2, \ldots$ where α_1, α_2 are consts.

Fwd if $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ then $\{a_n\}$ is sol.

$\Rightarrow$ b/c r_1, r_2 roots of $r^2 - c_1 r_2 - c_2 = 0 \Rightarrow \underline{r_1^2 = c_1 r_1 + c_2}$

$\underline{r_2^2 = c_1 r_2 + c_2}$

$$\Rightarrow c_1 a_{n-1} + c_2 a_{n-2} = c_1(\alpha_1 r_1^{n-1} + \alpha_2 r_2^{n-1}) + c_2(\alpha_1 r_1^{n-2} + \alpha_2 r_2^{n-2})$$
$$= \alpha_1 r_1^{n-2}(c_1 r_1 + c_2) + \alpha_2 r_2^{n-2}(c_1 r_2 + c_2)$$
$$= \alpha_1 r_1^{n-2} r_1^2 + \alpha_2 r_2^{n-2} r_2^2$$
$$= \alpha_1 r_1^n + \alpha_2 r_2^n$$
$$= a_n$$

$\Rightarrow \{a_n\}$ w/ $a_n = a_1 r_1^n + a_2 r_2^n$ is a sol

2nd

Every sol $\{a_n\}$ of RR

$$a_n = C_1 a_{n-1} + C_2 a_{n-2}$$

has $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ for $n = 0, 1, 2.$ for α_1, α_2 consts

Suppose $\{a_n\}$ is a sol with init cond. $a_0 = C_0$ $a_1 = C_1$ hold

We will show $\exists\, \alpha_1, \alpha_2$ s.t. $a_n = \alpha_1 r_1^n + \alpha_2 r_2^n$ satisfies initial conds

Need $\Rightarrow$ $a_0 = C_0 = \alpha_1 + \alpha_2$

$a_1 = C_1 = \alpha_1 r_1 + \alpha_2 r_2$

Solve for α_1, α_2

$\Rightarrow \alpha_2 = C_0 - \alpha_1$

$\Rightarrow C_1 = \alpha_1 r_1 + (C_0 - \alpha_1) r_2$

$= \alpha_1 (r_1 - r_2) + C_0 r_2$

$$\alpha_1 = \frac{C_1 - C_0 r_2}{r_1 - r_2}$$

$$\alpha_2 = C_0 - \alpha_1 = C_0 - \frac{C_1 - C_0 r_2}{r_1 - r_2} = \frac{C_0 r_1 - C_1}{r_1 - r_2}$$

(need $r_1 \neq r_2$)

Ex fib.

$f_n = f_{n-1} + f_{n-2}$ $\quad f_0 = 0 \quad f_1 = 1$

$r^2 - r - 1 = 0$

$r_1 = \dfrac{1+\sqrt{5}}{2} \qquad r_2 = \dfrac{1-\sqrt{5}}{2}$

by Thm 1 $\Rightarrow f_n = \alpha_1 \left(\dfrac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\dfrac{1-\sqrt{5}}{2}\right)^n$

$f_0 = \alpha_1 + \alpha_2 = 0$

$f_1 = \alpha_1 \left(\dfrac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\dfrac{1-\sqrt{5}}{2}\right) = 1$

$\alpha_1 = 1/\sqrt{5} \qquad \alpha_2 = -1/\sqrt{5}$

$f_n = \dfrac{1}{\sqrt{5}} \left(\dfrac{1+\sqrt{5}}{2}\right)^n + -\dfrac{1}{\sqrt{5}} \left(\dfrac{1-\sqrt{5}}{2}\right)^n$